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A Class of Two-Point Boundary Value Problems for Systems of Ordinary Differential Equations

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INTRODUCTION

This paper deals with a certain class of two-point boundary value problems of the form

$$\begin{aligned} &x' = f(t, x), \qquad x = (x^1, ..., x^n), \\ &x^i(t_1) = \alpha^i,\ i \in N; \qquad x^j(t_2) = \beta^j,\ j \in M, \end{aligned} \tag{$*$}$$

where $f\colon [t_1, t_2] \times R^n \to R^n$, and M and N are arbitrary not necessarily disjoint subsets of $\{1, ..., n\}$ such that card M + card $N = n$.

Among papers regarding multiple-point boundary value problems we wish to point out a group of papers by Garner [3–5] and by Garner and Barton [6]. In these papers two-point and three-point problems are investigated only for linear systems of differential equtions and nth order linear equations. The nature of sufficient conditions for uniqueness results is investigated subject to sign restriction on the off-diagonal entries in the matrix of a system and the coefficients of an differential equation.

Some results of [6] are generalized in the author's paper [7].

Problems similar to those considered here have been discussed by Urbanovič [11] for linear systems and by Čurikov *et al.* [1] for 3rd and 4th order differential equations.

There are not many papers going in such a direction.

The reader interested in other results regarding boundary value problems and related topics is referred to the extensive bibliography contained in the article by Gingold [8].

The purpose of the present paper is to show that results analogous to those mentioned before may be extended to all two-point boundary value problems of the form $(*)$, where the restrictions on the signs refer to the entries in the Jacobian matrix of f.

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Our results concerning the existence and uniqueness of the solution to $(*)$ are a natural extension of those from [1, 3, 6, 7].

The paper is divided into parts devoted to linear and nonlinear systems. In the first part (Sections 1–4) we give the notations, introduce the basic definitions and state some auxiliary lemmas. The end of the first part (Section 4) contains our main theorem for the linear systems.

In the second part (Sections 5–6) we extend the results from Part I to the case of the nonlinear systems.

I. Linear Systems

1. We start with certain notations and definitions.

Let $R_{n\times n}$ be the set of all real $n\times n$ matrices. By $\Lambda_{n\times n}$ we denote the subset of $R_{n\times n}$ consisting of all matrices of the form

$$\Lambda=\lambda\lambda^T,$$

where λ is an n-column vector with coordinates $|\lambda^i|=1$, $i=1,..., n$, and λ^T its transpose. We note that an arbitrary matrix in $\Lambda_{n\times n}$ has all entries *equal* to 1 or -1. Moreover, from the above definition, it follows that the set $\Lambda_{n\times n}$ consists of exactly 2^{n-1} different symmetric matrices $\Lambda=(\lambda^i_j)$ with the property

$$\lambda^i_k\lambda^k_j=\lambda^i_j \qquad \text{for } i, j, k=1,..., n. \tag{1.1}$$

If $A=(a^i_j)$ and $B=(b^i_j)$ are real $n\times n$ matrices we use the symbol $A\circ B$ to denote the *Hadamard product*: $A\circ B=(a^i_jb^i_j)$, $i, j=1,..., n$.

The notation $A\geqslant 0$ $(A>0)$ indicates that all entries of A are *nonnegative* (*positive*). However, we use the notation $A\geqslant B$ $(A>B)$ if $A-B\geqslant 0$ $(A-B>0)$.

Let $J=[t_1, t_2](t_1<t_2)$ be a compact interval of the real line R. Then by $C_{n\times n}(J)$ we understand the class of all real $n\times n$ matrices whose entries are continuous functions on J. The interval $J=[t_1, t_2]$ will be fixed throughout this paper, and we abbreviate $C_{n\times n}(J)$ to $C_{n\times n}$.

Now for any given $\Lambda\in\Lambda_{n\times n}$, the symbol $\mathscr{D}_\Lambda$ denotes the class of all matrices $A\in C_{n\times n}$ with the property that

$$\Lambda\circ A(t)\geqslant 0 \qquad \text{for all } t\in J.$$

This property means that the sign of non-zero entry a^i_j of A is equal to that of λ^i_j. Moreover, it is clear that $\mathscr{D}_\Lambda$ is nonempty. The class $\mathscr{D}_\Lambda$ plays an important role in our paper.

Further on let Λ be arbitrary but fixed in $\Lambda_{n\times n}$.

Now we prove two lemmas which we shall use in the sequel.

LEMMA 1.1. *The following basic properties of the class hold:*

(i) *If* $A, B \in \mathscr{D}_\Lambda$, *then* $A + B$, $AB \in \mathscr{D}_\Lambda$.

(ii) *If* $A_n \in \mathscr{D}_\Lambda$, $n = 1, 2, \ldots$, *and* $\lim_{n \to \infty} A_n = A \in C_{n \times n}$, *then so is* A.

The proof of (ii) and of the first part of (i) is trivial.

The condition $AB \in \mathscr{D}_\Lambda$ follows at once from the identity

$$(\Lambda \circ A)(\Lambda \circ B) = \Lambda \circ (AB) \tag{1.2}$$

which is easily derived from the relationship (1.1).

LEMMA 1.2. *Let* $A \in \mathscr{D}_\Lambda$ *and* $\{A_k\}$ *be a sequence of matrices determined by* A *as follows:*

$$\begin{aligned} A_1(t) &= I_n, \\ A_k(t) &= \int_{t_1}^{t} A(\tau)\, A_{k-1}(\tau)\, d\tau \qquad \text{for } t \in J,\ k \geqslant 2, \end{aligned} \tag{1.3}$$

where I_n *is the* $n \times n$ *identity matrix. Then* $\{A_k\}$ *belongs to* $\mathscr{D}_\Lambda$.

The proof of the above lemma uses the identity (1.2) and standard induction; it will be omitted.

2. For any given matrix A in $C_{n \times n}$ we denote by X_A the *fundamental matrix* of the system

$$x' = Ax,$$

i.e., the solution of the matrix system

$$X' = AX, \qquad X(t_1) = I_n. \tag{2.1}$$

Obviously $X_A \in C_{n \times n}$. Furthermore, it follows from well-known results that X_A is given by the infinite series

$$X_A = \sum_{k=1}^{\infty} A_k, \tag{2.2}$$

where A_k is defined by (1.3).

From (2.2) and Lemmas 1.1 and 1.2 we get immediately the following lemma

LEMMA 2.1. *Let* A *be an arbitrary matrix of* $\Delta_{n \times n}$. *If* A *belongs to* $\mathscr{D}_\Lambda$, *then* X_A *also has this property.*

Before stating our next result of this section we introduce the following notations.

For any matrix $A \in C_{n\times n}$ with entries a_j^i, $i, j = 1,..., n$, define

$$A^0 = A - \operatorname{diag}(a_1^1,..., a_n^n),$$

and the diagonal matrix E_A as

$$E_A(t) = \operatorname{diag}\left(\exp\int_{t_1}^{t} a_1^1(\tau)\,d\tau,..., \exp\int_{t_1}^{t} a_n^n(\tau)\,d\tau\right).$$

For later purposes, note that the matrix E_A is invertible and

$$E_A,\ E_A^{-1} \in \mathscr{D}_A$$

for any $A \in \Delta_{n\times n}$.

We now generalize Lemma 2.1 as follows

LEMMA 2.2. *If matrix A^0 belongs to $\mathscr{D}_A$, then X_A belongs to the same class as A^0.*

In addition the following estimate holds on J

$$\begin{aligned} \Lambda \circ X_A(t) \geqslant E_A(t)\Bigg\{I_n + \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}} \\ \times \prod_{s=1}^{l-1} E_A^{-1}(\tau_s)\,(\Lambda \circ A^0(\tau_s))\,E_A(\tau_s)\,d\tau_{l-1}\Bigg\} \\ \geqslant E_A(t)\ (\tau_0 = t), \end{aligned} \tag{2.3}$$

for each integer l, $l \geqslant 2$.

Proof. Let B be the matrix defined as the product of matrices E_A^{-1}, A^0 and E_A, i.e.,

$$B = E_A^{-1} A^0 E_A.$$

Since E_A^{-1}, A^0 and E_A are matrices of the class $\mathscr{D}_A$ we infer by Lemma 1.1(i) that B also has this property. Thus, by virtue of the previous lemma, it follows that the fundamental matrix for B, X_B belongs to $\mathscr{D}_A$. Inasmuch as X_B and X_A satisfy on J the relation

$$E_A X_B = X_A \tag{2.4}$$

we conclude that X_A also belongs to $\mathscr{D}_A$. The proof of the first part of this lemma is therefore complete.

Now let $\{B_i\}$ be the sequence of matrices defined by (1.3) with A replaced by B, i.e.,

$$B_1(t) = I_n,$$

$$B_i(t) = \int_{t_1}^{t} B(\tau)\, B_{i-1}(\tau)\, d\tau$$

$$= \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2, \ldots, \int_{t_1}^{\tau_{i-2}} \prod_{s=1}^{i-1} E_A^{-1}(\tau_s)\, A^0(\tau_s)\, E_A(\tau_s)\, d\tau_{i-1},$$

$$i \geqslant 2 \; (\tau_0 = t).$$

Then $X_B = \sum_{i=1}^{\infty} B_i$, where $B_i \in \mathscr{D}_A$, by Lemma 1.2. Since

$$X_B - (I_n + B_l) = \sum_{\substack{i=1 \\ i \neq l}}^{\infty} B_i,$$

for arbitrary l, $l \geqslant 2$, and, as follows from Lemma 1.1, the right-hand side of this equality is the matrix of the class $\mathscr{D}_A$, so we obtain

$$\Lambda \circ (X_B - (I_n + B_l)) \geqslant 0, \qquad l \geqslant 2.$$

Therefore

$$\Lambda \circ (E_A^{-1} X_A) \geqslant I_n + \Lambda \circ B_l \geqslant I_n, \qquad l \geqslant 2,$$

follows from (2.4), the equality $\Lambda \circ I_n = I_n$, and the fact $\Lambda \circ B_l \geqslant 0$. On the other hand, using (1.2) and the obvious equality $E_A^{-1} \circ \Lambda = E_A^{-1}$, we get

$$\Lambda \circ (E_A^{-1} X_A) = E_A^{-1}(\Lambda \circ X_A)$$

and

$$\Lambda \circ B_l(t) = \Lambda \circ \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2, \ldots, \int_{t_1}^{\tau_{l-2}}$$

$$\times \prod_{s=1}^{l-1} E_A^{-1}(\tau_s)\, A^0(\tau_s)\, E_A(\tau_s)\, d\tau_{l-1}$$

$$= \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2, \ldots, \int_{t_1}^{\tau_{l-2}}$$

$$\times \prod_{s=1}^{l-1} E_A^{-1}(\tau_s)(\Lambda \circ A^0(\tau_s))\, E_A(\tau_s)\, d\tau_{l-1}, \qquad l \geqslant 2 \; (\tau_0 = t).$$

Thus the last estimate takes the form

$$E_A^{-1}(t)(A \circ X_A(t)) \geqslant I_n + \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}}$$
$$\times \prod_{s=1}^{l-1} E_A^{-1}(\tau_s)(A \circ A^0(\tau_s))\, E_A(\tau_s)\, d\tau_{l-1}$$
$$\geqslant I_n.$$

This, together with the fact the diagonal entries of E_A are positive, implies the desired estimate (2.3). Hence the proof is complete.

For subsequent use, we make the following corollary.

COROLLARY 2.1. *For the entries of the fundamental matrix for A, $X_A = (x_{A\,j}^{\,i})$ the following estimates hold on J*

$$\lambda_j^i x_{A\,j}^{\,i}(t) \geqslant \exp \int_{t_1}^{t} a_i^i(\tau)\, d\tau$$
$$\times \left\{ \delta_j^i + \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}} \right.$$
$$\times \left(\prod_{s=0}^{l-2} \lambda_{i_{s+1}}^{i_s} a_{i_{s+1}}^{i_s}(\tau_{s+1}) \right.$$
$$\left. \times \exp \int_{t_1}^{\tau_{s+1}} (a_{i_{s+1}}^{i_{s+1}}(\tau) - a_{i_s}^{i_s}(\tau))\, d\tau \right) d\tau_{l-1} \Bigg\}$$
$$\geqslant \delta_j^i \exp \int_{t_1}^{t} a_i^i(\tau)\, d\tau, \qquad i, j = 1,..., n, \tag{2.5}$$

where $\{i_0, i_1,..., i_{l-1}\}$ is a finite subset of $\{1,..., n\}$ such that

$$i_s \neq i_{s+1} \qquad \text{for } s = 0, 1,..., l-2,\ i_0 = i,\ i_{l-1} = j, \tag{2.6}$$

and δ_j^i is the Kronecker delta.

Proof. This is an immediate consequence of (2.3).

In the sequel we shall need the following additional definition.

We shall say that a matrix $A = (a_j^i) \in C_{n \times n}$ is *irreducible on J with respect to the pair of indices (i, j), $i \neq j$* (ith row and jth column) if there exists a finite sequence $i_0, i_1,..., i_{l-1}$ of l positive integers of the set $\{1,..., n\}$ satisfying (2.6) and there exists a non-increasing sequence $\xi_1, \xi_2,..., \xi_{l-1}$ of points from J such that

$$a_{i_{s+1}}^{i_s}(\xi_{s+1}) \neq 0 \qquad \text{for } s = 0, 1,..., l-2. \tag{2.7}$$

Clearly it is sufficient to consider $l \leqslant n$. The statement that a matrix $A \in C_{n \times n}$ is *irreducible*, means that A is irreducible with respect to each pair of indices (i, j), $i \neq j$, $i, j = 1, \ldots, n$.

Remark. Let $A \in C_{n \times n}$. If there exists $t_0 \in J$ such that the constant matrix $A(t_0)$ is irreducible (in the classical sense, see [2]), then the matrix A is irreducible on J.

Moreover we notice that if A is irreducible, then A^0 has the same property.

Also the following remark is true.

Remark. If a matrix A belongs $\mathscr{D}_\Lambda$, then the condition (2.7) is equivalent to the inequality

$$\prod_{s=0}^{l-2} \lambda^{i_s}_{i_{s+1}} a^{i_s}_{i_{s+1}}(\xi_{s+1}) > 0, \tag{2.8}$$

where λ^i_j, $i, j = 1, \ldots, n$, are the entries of Λ.

We conclude this section with the following simple observation.

COROLLARY 2.2. *If in addition to the hypotheses of Lemma 2.2, A is irreducible, then we have*

$$\lambda^i_j x_A{}^i_j(t_2) > 0, \qquad \text{for } i, j = 1, \ldots, n, \tag{2.9}$$

where λ^i_j and $x_A{}^i_j$ are entries of matrices Λ and X_A, respectively.

Indeed, it follows from the estimate (2.5) in Corollary 2.1 and the inequality (2.8).

In particular, the system of estimates (2.9) can be equivalently written in the matrix form

$$\Lambda \circ X_A(t_2) > 0. \tag{2.10}$$

3. In the sequel we shall need the following additional definitions.

Let r be an arbitrary positive integer, $1 \leqslant r \leqslant n$. We denote by $G_{r,n}$ the class of all *strictly increasing* sequences of r positive integers of the set $\{1, \ldots, n\}$. We introduce in $G_{r,n}$ lexicographic order. Now with each element $\alpha \in G_{r,n}$ we associate the positive integer h_α as follows: h_α denotes the *place* of α in our order. Obviously h_α is uniquely defined by $\alpha \in G_{r,n}$, and each positive integer h such that $1 \leqslant h \leqslant \binom{n}{r}$ unique defines the element $\alpha \in G_{r,n}$ for which $h_\alpha = h$.

If $A = (a^i_j)$ is a given $n \times n$ matrix and $\alpha = (i_1, \ldots, i_r)$, $\beta = (j_1, \ldots, j_r)$ are two systems of r positive integers such that $1 \leqslant i_s, j_s \leqslant n$ for $s = 1, \ldots, r$, we set

$$A\begin{pmatrix} i_1, \ldots, i_r \\ j_1, \ldots, j_r \end{pmatrix} = \det(a^{i_l}_{j_k}), \qquad l, k = 1, \ldots, r.$$

We shall occasionally write $A\binom{\alpha}{\beta}$ instead of

$$A\begin{pmatrix} i_1,\dots,i_r \\ j_1,\dots,j_r \end{pmatrix}.$$

By $M_r(A)$ we shall denote the $\binom{n}{r}\times\binom{n}{r}$ matrix with the entries $m^i_j(A)$ defined as

$$m^i_j(A)=A\begin{pmatrix}\alpha\\ \beta\end{pmatrix},$$

where $\alpha,\ \beta\in G_{r,n}$ are such that $h_\alpha=i,\ h_\beta=j$.

In order to define the next $\binom{n}{r}\times\binom{n}{r}$ matrix we introduce the symbols $b^\alpha_\beta(A)$, $\alpha,\ \beta\in G_{r,n}$, $\alpha=(i_1,\dots,i_r)$ and $\beta=(j_1,\dots,j_r)$ as follows:

$$\begin{aligned} b^\alpha_\beta(A) &= b\begin{pmatrix} i_1,\dots,i_r \\ j_1,\dots,j_r \end{pmatrix}(A) \\ &= \sum_{s=1}^{r} a^{i_s}_{i_s} && \text{if } \alpha=\beta=(i_1,\dots,i_r) \\ &= (-1)^{p+q}\, a^{i_p}_{j_q} && \text{if } \alpha\backslash\beta=\{i_p\} \text{ and } \beta\backslash\alpha=\{j_q\} \\ &= 0 && \text{otherwise.} \end{aligned} \tag{3.1}$$

Then by $B_r(A)=(b^i_j(A))$, we understand the $\binom{n}{r}\times\binom{n}{r}$ matrix for which $b^i_j(A)=b^\alpha_\beta(A)$, where $\alpha,\ \beta\in G_{r,n}$ are defined by the conditions $h_\alpha=i,\ h_\beta=j$. The matrix $B_r(A)$ is very useful in our discussion.

From the definition of $B_r(A)$, it follows that in the cases $r=1$ and $r=n$ we have $B_1(A)=A$ and $B_n(A)=\operatorname{tr} A$, respectively, where $\operatorname{tr} A$ denotes the trace of A. However, in the case $r=n-1$, i.e., for the matrix $B_{n-1}(A)$ we obtain

$$b^i_j(A)=(-1)^{i+j+1}\, a^{n+1-j}_{n+1-i}, \qquad 1\leqslant i,j\leqslant n,\ i\neq j,$$

$$b^i_i(A)=\sum_{\substack{s=1\\ s\neq n+1-i}}^{n} a^s_s, \qquad 1\leqslant i\leqslant n,$$

where $A=(a^i_j)$.

Furthermore,

$$B^0_r(A)=B_r(A^0), \qquad 1\leqslant r\leqslant n.$$

At this point we can state the following lemma.

LEMMA 3.1. *Let* $A \in C_{n \times n}$. *Then* $B_r(A) \in C_{N \times N}$, *where* $N = \binom{n}{r}$, *and*

$$X_{B_r(A)} = M_r(X_A) \qquad \text{for each } r = 1,\dots, n, \tag{3.2}$$

i.e., the matrix $M_r(X_A)$ *is the solution of* (2.1), *where* A *is replaced by* $B_r(A)$.

For a proof of this lemma the reader is referred to Schwarz [10, Theorem 1].

Now we may combine Lemma 3.1, Lemma 2.2 and its Corollary 2.2 to obtain the following corollary

COROLLARY 3.1. *Let* $r \in \{1,\dots, n\}$ *and let* $\Lambda \in \Delta_{N \times N}$, *where* $N = \binom{n}{r}$. *Suppose that* $A \in C_{n \times n}$ *is such that*

$$B_r^0(A) \in \mathscr{D}_\Lambda.$$

Then we have

$$\Lambda \circ M_r(X_A(t_2)) \geq E_{B_r(A)}(t_2). \tag{3.3}$$

If in addition $B_r(A)$ *is irreducible, then we have*

$$\Lambda \circ M_r(X_A(t_2)) > 0. \tag{3.4}$$

Indeed, if in Lemma 2.2 we replace A^0 by $B_r^0(A)$, then by virtue of the estimate (2.3) at $t = t_2$ and by the equality (3.2), we get (3.3). On the other hand, the last inequality follows from (2.10) and (3.2).

4. We shall consider the linear nonhomogeneous system of differential equations

$$x' = A(t)\, x + g(t), \qquad t \in J = [t_1, t_2], \tag{4.1}$$

where $A \in C_{n \times n}$ and $g \in C_n$, the class of all continuous on J n-column vector functions.

Let P_1, P_2 be $n \times n$ real matrices. We shall be interested in the existence and uniqueness of solutions of (4.1) satisfying

$$P_1 x(t_1) + P_2 x(t_2) = p \tag{4.2}$$

for a given $p \in R^n$.

The following result is well known (e.g., see [8, 10]).

LEMMA 4.1. *The boundary value problem* (4.1), (4.2) *has a unique solution for every* $p \in R^n$ *and every* $g \in C_n$ *if the matrix* $(P_1 + P_2 X_A(t_2))$ *is nonsingular.*

This solution is determined by the initial condition of the form

$$x_0 = x(t_1) = (P_1 + P_2 X_A(t_2))^{-1} (p - P_2 v(t_2)), \tag{4.3}$$

where

$$v(t) = \int_{t_1}^{t} X_A(t)\, X_A^{-1}(\tau)\, g(\tau)\, d\tau.$$

In the sequel we shall consider the problem (4.1), (4.2) with a special form of the matrices in (4.2).

For this purpose we shall need the following notations.

Let I_q^p, $p, q = 1,\dots, n$, denote the matrices from $R_{n\times n}$ whose all entries equal zero, except the entry in the (p, q) place which equals one, i.e., $I_q^p = (\delta_i^q \delta_p^j)$, $i, j = 1,\dots, n$. If $\alpha = (i_1,\dots, i_k)$ and $\beta = (j_1,\dots, j_k)$ are sequences of the class $G_{k,n}$, $1 \leqslant k \leqslant n$, we use the symbol I_β^α to denote the $n\times n$ matrix defined as

$$I_\beta^\alpha = I_{j_1}^{i_1} + \cdots + I_{j_k}^{i_k}.$$

Moreover, with each sequence $\alpha \in G_{k,n}$ we associate the sequence $\bar\alpha$ from $G_{n-k,n}$ such that

$$\alpha \cup \bar\alpha = \{1,\dots, n\}.$$

Now we put

$$P_1 = I_\alpha^\alpha \qquad \text{and} \qquad P_2 = I_\beta^{\bar\alpha},$$

where α and β are sequences from $G_{k,n}$ and $G_{n-k,n}$ $(1 \leqslant k \leqslant n-1)$, respectively. Thus the boundary condition (4.2) takes the form

$$I_\alpha^\alpha x(t_1) + I_\beta^{\bar\alpha} x(t_2) = p. \tag{4.4}$$

It is easy to see that any two sequences $\alpha = (i_1,\dots, i_k)$ and $\beta = (j_1,\dots, j_{n-k})$—possibly coincident—assigned the respective sets of coordinates of a solution x of (4.1) at the points t_1 and t_2 for which

$$x^{i_s}(t_1) = p^{i_s}, \qquad s = 1,\dots, k,$$
$$x^{j_s}(t_2) = p^{\bar i_s}, \qquad s = 1,\dots, n-k,$$

where $\bar\alpha = (\bar i_1,\dots, \bar i_{n-k})$ and $p = (p^1,\dots, p^n) \in R^n$(see $(*)$).

We state here the following result, which is a direct consequence of Lemma 4.1.

COROLLARY 4.1. *Let us the problem* (4.1), (4.4) *be given. If*

$$X_A\left(t_2; \frac{\beta}{\bar\alpha}\right) \neq 0, \tag{4.5}$$

then for each $p \in R^n$ and every $g \in C_n$ the boundary value problem (4.1), (4.4) *has exactly one solution.*

Proof. The reader can verify using the well-known expansion by the elements of columns that

$$\det(I^{\alpha}_{\alpha} + I^{\bar{\alpha}}_{\beta} X_A(t_2)) = X_A\left(t_2; \begin{matrix}\beta \\ \bar{\alpha}\end{matrix}\right).$$

Thus (4.5) is an immediate consequence of Lemma 4.1.

Our main result of this section is the followng theorem.

THEOREM 4.1. *Let k, with $1 \leqslant k \leqslant n-1$, be given. Assume that the system* (4.1) *satisfies the following conditions*:

(i) there exists a matrix $\Lambda \in \Delta_{N \times N}$, where $N = \binom{n}{n-k}$ such that $B^0_{n-k}(A) \in \mathscr{D}_\Lambda$,

(ii) *the matrix $B^0_{n-k}(A)$ is irreducible.*

Then for each $\alpha \in G_{k,n}$, $\beta \in G_{n-k,n}$ and $p \in R^n$ there exists exactly one solution of (4.1), (4.4).

Proof. To prove the theorem, it suffices to verify condition (4.5) of Corollary 4.1.

Let $\alpha \in G_{k,n}$ and $\beta \in G_{n-k,n}$ be fixed and Λ be a matrix given in (i).

In order to show that (4.5) is true we first observe that the matrix $B_{n-k}(A)$ satisfies the assumptions of Corollary 3.1 with $r = n-k$. Therefore the estimate (3.4) in Corollary 3.1 implies that

$$\Lambda \circ M_{n-k}(X_A(t_2)) > 0.$$

Hence, in particular, by the definition of $M_{n-k}(X_A)$ we have $X_A(t_2; \begin{smallmatrix}\beta \\ \bar{\alpha}\end{smallmatrix})\, \lambda^{h_\beta}_{h_{\bar{\alpha}}} > 0$, which assures that $X_A(t_2; \begin{smallmatrix}\beta \\ \bar{\alpha}\end{smallmatrix}) \neq 0$ since $\lambda^{h_\beta}_{h_{\bar{\alpha}}} \neq 0$. Thus (4.5) holds and the proof of Theorem 4.1 is finished.

Remark. In particular, we may take $\beta = \bar{\alpha}$. Then the conclusion of Theorem 4.1 remains true without condition (ii).

The proof proceeds along similar lines as above, but we use (3.3) instead of (3.4).

Now we shall analyse Theorem 4.1 in some special cases.

EXAMPLE 4.1. Consider the nth order differential equation

$$x^{(n)} = a_{n-1}(t)\, x^{(n-1)} + \cdots + a_0(t)\, x + u(t), \qquad x \in R, \tag{4.6}$$

with the boundary value conditions

$$x^{(m)}(t_1)=c^m, \qquad x^{(p)}(t_2)=c^p, \qquad p=0, 1,..., n-1, p\neq m. \tag{4.7}$$

Assume that the functions a_s, $s=0, 1,..., n-1$; u are continuous on $J=[t_1, t_2]$. Assume further that the inequality

$$(-1)^{n-s}\, a_s(t)\geqslant 0, \qquad t\in J, \tag{4.8}$$

holds for $s=0, 1,..., n-2$. Using Theorem 4.1 and its remark we shall show that for arbitrary m, $0\leqslant m\leqslant n-1$, the boundary value problem (4.6), (4.7) admits exactly one solution. In fact, (4.6), (4.7) are equivalent to (4.1), (4.4), respectively, with $A=(a^i_j)$, where $a^i_{i+1}=1$, $a^n_i=a_{i-1}$, $i=1,..., n$, and zero elsewhere, $g=(0,..., 0, u)\in C_n$, and $\alpha=(m)\in G_{1,n}$, $\beta=\bar{\alpha}\in G_{n-1,n}$. By means of (4.8) and the definition of $B_{n-1}(A)$ it is easy to check that the matrix $B^0_{n-1}(A)$ belongs to the $\mathscr{D}_\Lambda$ class, where Λ is determined by the vector $\lambda=(1,..., 1)\in R^n$. Thus according to Theorem 4.1 and its remark, this implies that the problem (4.6), (4.7) has exactly one solution.

This fact is a generalization of a result of Garner [3, Theorem 1].

In the same way as above we obtain Corollaries 2, 7 and 8 of [1] from Theorem 4.1 special cases.

EXAMPLE 4.2. Consider the problem (4.1), (4.4) for $\alpha=(1,..., p-1, p+1,..., n)\in G_{n-1,n}$, $\beta=(q)\in G_{1,n}$. Assume that the entries a^i_j of A for i, $j=1,..., n-1$, $i\neq j$, satisfy on J the conditions

$$a^i_j a^i_n a^j_n>0, \qquad a^i_n a^n_i>0. \tag{4.9}$$

Note that the matrix $B_1(A)=A$ is irreducible since $a^i_j\neq 0$ on J. Moreover (4.9) implies that $A^0\in\mathscr{D}_\Lambda$ if Λ is defined by the vector $\lambda=(\operatorname{sgn} a^n_1,..., \operatorname{sgn} a^n_{n-1}, 1)$. From Theorem 4.1, it follows that the above problem has exactly one solution for arbitrary p and q.

This example demonstrates that Theorem 4.1 is an improvement over Theorem 1 in Garner and Burton [6] and its extension due to the author in [7].

II. NONLINEAR SYSTEMS

5. Consider now the system of nonlinear differential equations

$$x'=f(t, x), \tag{5.1}$$

where x and $f(t, x)$ are n-dimensional column vectors.

Assume that the right-hand side of (5.1) satisfies the following condition:

(A) The function $f(t, x) = (f^1(t, x), \ldots, f^n(t, x))$ and its Jacobian matrix $F(t, x) = (f^i_{x^j}(t, x))$, $i, j = 1, \ldots, n$, with respect to x are defined and continuous in the set $J \times R^n$, $J = [t_1, t_2]$.

We shall be concerned in this section with the existence and uniqueness of a solution of (5.1) subject to the boundary condition (4.2), i.e.,

$$P_1 x(t_1) + P_2 x(t_2) = p.$$

In the following discussion the symbol $\|\cdot\|$ will denote the sum-norm in R^n and the corresponding norm for $n \times n$ real matrices in $R_{n \times n}$.

One can easily prove the following two lemmas.

LEMMA 5.1. *Let $\mathscr{K}$ be a uniformly bounded family of $n \times n$ real matrices; that is, there exists an $v > 0$ such that for each $K \in \mathscr{K}$*

$$\|K\| \leqslant v.$$

Moreover, we assume that

$$|\det K| \geqslant \delta$$

for each K in $\mathscr{K}$, where δ is a positive constant. Then the family $\mathscr{K}^{-1} = \{K^{-1} : K \in \mathscr{K}\}$ also has the above properties.

LEMMA 5.2. *Let $\mathscr{A}$ be a set in $C_{n \times n}$ with the property that there exists a positive constant μ such that*

$$\|A(t)\| \leqslant \mu \qquad \text{on } J$$

for each $A \in \mathscr{A}$. Then the set $\{X_A : A \in \mathscr{A}\}$, i.e., the set of all fundamental matrices X_A, $A \in \mathscr{A}$, also has this property.

In the sequel we associate with a pair (x_1, x_2), where $x_1, x_2 \in C_n$, a *matrix* F_{x_1, x_2} defined by

$$F_{x_1, x_2}(t) = \int_0^1 F(t, s x_2(t) + (1 - s)\, x_1(t))\, ds.$$

It follows from (A) that the matrix F_{x_1, x_2} is continuous on J.

Also with a pair (x_1, x_2) we associate a *matrix* $D_{x_1, x_2} \in R_{n \times n}$, as follows:

$$D_{x_1, x_2} = P_1 + P_2 X_{F_{x_1, x_2}}(t_2).$$

We recall here $X_{F_{x_1, x_2}}$ is the fundametal matrix for F_{x_1, x_2}, i.e., the solution of the system (2.1) with respect to F_{x_1, x_2} in place of A.

We make the following corollary.

COROLLARY 5.1. *Suppose that all matrices, of the family* $\{F(t, x): x \in C_n\}$ *are uniformly bounded on* J, *i.e., there exists a positive constant* μ *such that*

$$\|F(t, x)\| \leqslant \mu \qquad \text{on } J \tag{5.2}$$

for every x *in* C_n. *Then the set* $\{X_{F_{x_1,x_2}}: x_1, x_2 \in C_n\}$ *also has this property.*

Proof. It follows from (5.2) that

$$\begin{aligned} \|F_{x_1,x_2}(t)\| &= \left\| \int_0^1 F(t, sx_2(t) + (1-s)\, x_1(t))\, ds \right\| \\ &\leqslant \int_0^1 \|F(t, sx_2(t) + (1-s)\, x_1(t))\|\, ds \\ &\leqslant \mu \qquad \text{on } J \end{aligned} \tag{5.3}$$

for every x_1, x_2 in C_n. An application of Lemma 5.2 with $\mathscr{A} = \{F_{x_1,x_2}: x_1, x_2 \in C_n\}$ terminates the proof.

At this point we can state the following lemma.

LEMMA 5.3. *Let a triple* $(f; P_1, P_2)$ *be given, where* $f\colon J \times R^n \to R^n$ *is a function satisfying condition* (A) *and* P_1, P_2 *are matrices from* $R_{n \times n}$.

Assume that condition (5.2) *is satisfied. In addition suppose that there exists a positive constant* δ *such that*

$$|\det D_{x_1,x_2}| = |\det(P_1 + P_2 X_{F_{x_1,x_2}}(t_2))| \geqslant \delta \tag{5.4}$$

for every x_1, x_2 *in* C_n. *Then there exists a neighbourhood* $\mathscr{U}_{P_1}$ *of* P_1 *such that for each* $S \in \mathscr{U}_{P_1}$ *and* $p \in R^n$ *there exists at most one solution of* (5.1) *subject to the boundary condition*

$$Sx(t_1) + P_2 x(t_2) = p. \tag{5.5}$$

Proof. First we obtain a bound on $D^{-1}_{x_1,x_2}$, x_1, $x_2 \in C_n$. From the definition of D_{x_1,x_2} and from Corollary 5.1 we get

$$\|D_{x_1,x_2}\| \leqslant \nu$$

for every x_1, x_2 in C_n, where ν is a positive constant. This, (5.4) and Lemma 5.1 imply that there exists an $\rho > 0$ such that

$$\|D^{-1}_{x_1,x_2}\| \leqslant \rho \tag{5.6}$$

for every $x_1, x_2 \in C_n$.

We now define $\mathscr{U}_{P_1}$ by

$$\mathscr{U}_{P_1} = \left\{ S \in R_{n\times n} : \| P_1 - S \| < \frac{1}{\rho} \right\}.$$

Suppose that the conclusion of the theorem with respect to $\mathscr{U}_{P_1}$ is false. Then for some $S \in \mathscr{U}_{P_1}$, $p \in R^n$, the boundary value problem (5.1), (5.5) has two distinct solutions: x_1, x_2. Their difference $x = x_1 - x_2$ by the Hadamard's lemma is a solution of the linear differential systems

$$x' = F_{x_1,x_2} x,$$

i.e., the system (4.1) with F_{x_1,x_2} in place of A and $g = 0$. Moreover this solution also satisfies the boundary condition

$$P_1 x(t_1) + P_2 x(t_2) = \bar{p},$$

where $\bar{p} = (P_1 - S)\, x(t_1)$. Therefore in view of (4.3) in Lemma 4.1 we get

$$x(t_1) = D_{x_1,x_2}^{-1} \bar{p},$$

so from the fact that $S \in \mathscr{U}_{P_1}$ and (5.6) we conclude

$$\begin{aligned} \|\bar{p}\| &= \|(P_1 - S)\, x(t_1)\| \\ &\leqslant \|P_1 - S\| \, \|x(t_1)\| \\ &< \|\bar{p}\|, \end{aligned}$$

which is a contradiction. This completes the proof.

Remark. Let $f: J \times R^n \to R^n$ be a function satisfying condition (A) and P_1, P_2 be matrices from $R_{n\times n}$. If for a triple $(f; P_1, P_2)$ the inequality

$$|\det D_{x_1,x_2}| > 0$$

holds for every x_1, x_2 in C_n, then the last theorem reduces to the uniqueness theorem of the problem (5.1), (4.2) proved by Gingold in [8, Proposition 4.2].

Before stating our main results of this section we shall need the following fact.

THEOREM. *Let a triple $(f; \mathscr{P}_1, \mathscr{P}_2)$ be given, where $\mathscr{P}_1$, $\mathscr{P}_2$ are sets of $n \times n$ matrices, and $f: J \times R^n \to R^n$ is a continuous function satisfying the following condition:*

(C) *For each point $(t_0, p_0) \in J \times R^n$ there exists exactly one solution of* (5.1), *defined on the whole J and such that $x(t_0) = p_0$.*

Let at least one of the sets $\mathscr{P}_1$, $\mathscr{P}_2$ be open.

Assume that for each $P_1 \in \mathscr{P}_1$, $P_2 \in \mathscr{P}_2$ and $p \in R^n$ there exists at most one solution of the boundary value problem (5.1), (4.2). *Then for each $P_1 \in \mathscr{P}_1$, $P_2 \in \mathscr{P}_2$ and $p \in R^n$ there exists one solution of* (5.1), (4.2).

For a proof of this theorem the reader is referred to Lasota [9, Theorem 1].

Combining the theorem of Lasota and Lemma 5.3 we can obtain our main result relative to the boundary value problem (5.1), (4.2).

THEOREM 5.1. *Let a triple $(f; P_1, P_2)$ be given, where $f\colon J \times R^n \to R^n$ is a function satisfying condition* (A) *and P_1, P_2 are matrices from $R_{n \times n}$.*

Under the same hypotheses as in Lemma 5.3, *there exists one solution of* (5.1), (4.2) *for each $p \in R^n$.*

Proof. We put

$$\mathscr{P}_1 = \mathscr{U}_{P_1}, \qquad \mathscr{P}_2 = \{P_2\},$$

where $\mathscr{U}_{P_1}$ is defined in Lemma 5.3. By virtue of the last lemma the triple $(f; \mathscr{P}_1, \mathscr{P}_2)$ satisfies all the hypotheses of Lasota's theorem except condition (C), which follows from assumption (A) and the estimate (5.2). The proof now follows immediately from the theorem of Lasota.

6. In this section we apply the results of Section 5 to the case in which the boundary condition (4.2) takes the form (4.4). That is, consider the system (5.1) subject to the boundary condition (4.4).

Let Λ be a given matrix in $\Delta_{n \times n}$. A family $\mathscr{K}$ of matrices of the class $\mathscr{D}_\Lambda$ is said to be *irreducible* on J if there exists an irreducible matrix U in $\mathscr{D}_\Lambda$ such that

$$\Lambda \circ K^0 \geqslant \Lambda \circ U^0$$

for every K in $\mathscr{K}$. A matrix U is called a *minor-matrix* of $\mathscr{K}$.

Remark. Let $K \in \mathscr{D}_\Lambda$ be given. The family $\mathscr{K} = \{K\}$ is irreducible if and only if so is K.

The main result of this paper is:

THEOREM 6.1. *Let $f\colon J \times R^n \to R^n$ be a fixed function satisfying condition* (A), *and let the hypothesis* (5.2) *of Corollary* 5.1 *be satisfied.*

Moreover, assume that for a fixed integer k, where $1 \leqslant k \leqslant n-1$, the following conditions hold:

(i) *there exists a matrix* $\Lambda = (\lambda_j^i)$ *of the class* $\Delta_{N\times N}$, *where* $N = \binom{n}{n-k}$ *such that*

$$B_{n-k}^0(F(t, x)) \in \mathscr{D}_\Lambda$$

for every x *in* C_n;

(ii) *the family* $\{B_{n-k}(F(t, x)) \colon x \in C_n\}$ *is irreducible.*

Then for each $\alpha \in G_{k,n}$, $\beta \in G_{n-k,n}$ *and* $p \in R^n$, *there exists exactly one solution of* (5.1), (4.4).

Before proceeding to the proof of Theorem 6.1 we prove the following lemma.

LEMMA 6.1. *If* $\{B_{n-k}(F(t, x)) \colon x \in C_n\}$ *satisfies conditions* (i), (ii) *of Theorem* 6.1, *then* $\{B_{n-k}(F_{x_1,x_2}(t)) \colon x_1, x_2 \in C_n\}$ *also has this property.*

Proof. Let Λ be a matrix given in (i) and let U be a minor-matrix of $\{B_{n-k}(F(t, x)) \colon x \in C_n\}$. By virtue of our assumptions we have

$$\Lambda \circ B_{n-k}(F^0(t, sx_2(t) + (1-s)\, x_1(t))) \geqslant \Lambda \circ U^0(t)$$

for $t \in J$, $s \in [0, 1]$ and $x_1, x_2 \in C_n$. This together with the obvious equality

$$B_{n-k}(F^0_{x_1,x_2}(t)) = B_{n-k}\left(\int_0^1 F^0(t, sx_2(t) + (1-s)\, x_1(t))\, ds\right)$$

$$= \int_0^1 B_{n-k}(F^0(t, sx_2(t) + (1-s)\, x_1(t)))\, ds$$

implies

$$\Lambda \circ B_{n-k}(F^0_{x_1,x_2}(t)) \geqslant \Lambda \circ U^0(t) \geqslant 0$$

for $t \in J$, which proves the lemma.

Remark. If $\{B_{n-k}(F(t, x)) \colon x \in C_n\}$ satisfies only condition (i), then so does $\{B_{n-k}(F_{x_1,x_2}(t)) \colon x_1, x_2 \in C_n\}$. It follows at once from the last equality.

We are now ready to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Let $\alpha \in G_{k,n}$ and $\beta \in G_{n-k,n}$ be fixed. In view of Theorem 5.1 it is enough to show that the assumption (5.4) holds with P_1 and P_2 replaced by I_α^α and $I_\beta^{\bar{\alpha}}$, respectively. Thus it suffices to verify that

$$|\det(I_\alpha^\alpha + I_\beta^{\bar{\alpha}} X_{F_{x_1,x_2}}(t_2))| \geqslant \delta > 0 \tag{6.1}$$

for every x_1, x_2 in C_n, where as before $X_{F_{x_1,x_2}}$ denotes the fundamental matrix for F_{x_1,x_2}.

Let x_1, $x_2 \in C_n$ be fixed. From the last remark we obtain that the assumptions of Lemma 2.2 hold with $A = B_{n-k}(F_{x_1,x_2})$ and $\Lambda = (\lambda_j^i)$ as in (i). Further, in view of Lemma 3.1 we have

$$X_{B_{n-k}(F_{x_1,x_2})} = M_{n-k}(X_{F_{x_1,x_2}}),$$

where $X_{B_{n-k}(F_{x_1,x_2})}$ is the fundamental matrix for $B_{n-k}(F_{x_1,x_2})$. So by virtue of Corollary 2.1 given after Lemma 2.2 we obtain for the entry $X_{F_{x_1,x_2}}\binom{\beta}{\bar{\alpha}}$ of the matrix $M_{n-k}(X_{F_{x_1,x_2}})$ the estimate

$$\begin{aligned}\lambda_j^i X_{F_{x_1,x_2}}\left(t_2; \frac{\beta}{\bar{\alpha}}\right) \geqslant \exp \int_{t_1}^{t_2} b_i^i(F_{x_1,x_2}(\tau))\, d\tau \\ \times \left\{\delta_j^i + \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}} \left(\prod_{s=0}^{l-2} \lambda_{i_{s+1}}^{i_{s+1}} b_{i_{s+1}}^{i_{s+1}}(F_{x_1,x_2}(\tau_{s+1}))\right.\right. \\ \left.\left.\times \exp \int_{t_1}^{\tau_{s+1}} b_{i_{s+1}}^{i_{s+1}}(F_{x_1,x_2}(\tau)) - b_{i_s}^{i_s}(F_{x_1,x_2}(\tau))\, d\tau\right) d\tau_{l-1}\right\} \\ (i = i_0,\ i_{l-1} = j),\end{aligned} \tag{6.2}$$

where i, j are uniquely determined by the equations $i = h_\beta$, $j = h_{\bar{\alpha}}$. Using the fact that $\|F_{x_1,x_2}\| \leqslant \mu$ (see (5.3)) we deduce from the definition of $B_{n-k}(F_{x_1,x_2}) = (b_j^i(F_{x_1,x_2}))$ that

$$\exp \int_{t_1}^{t_2} b_i^i(F_{x_1,x_2}(\tau))\, d\tau \geqslant \exp(-(n-k)(t_2 - t_1)\,\mu) \tag{6.3}$$

and

$$\begin{aligned}\exp \int_{t_1}^{\tau_{s+1}} b_{i_{s+1}}^{i_{s+1}}(F_{x_1,x_2}(\tau)) - b_{i_s}^{i_s}(F_{x_1,x_2}(\tau))\, d\tau \\ \geqslant \exp(-2(n-k)(t_2 - t_1)\,\mu).\end{aligned} \tag{6.4}$$

Now let $U = (u_j^i) \in \mathscr{D}_\Lambda$ be a minor-matrix of the family $\{B_{n-k}(F_{x_1,x_2})$: $x_1, x_2 \in C_n\}$. In view of (6.2), the definition of U together with (6.3), (6.4) implies that

$$\begin{aligned}\lambda_j^i X_{F_{x_1,x_2}}\left(t_2; \frac{\beta}{\bar{\alpha}}\right) \\ \geqslant \exp(-\mu(n-k)(t_2 - t_1))\left\{\delta_j^i\right. \\ + \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}} \left(\prod_{s=0}^{l-2} \lambda_{i_{s+1}}^{i_s} u_{i_{s+1}}^{i_s}(\tau_{s+1})\right. \\ \left.\left.\times \exp 2(k-n)(l-1))\,\mu\right) d\tau_{l-1}\right\}.\end{aligned} \tag{6.5}$$

From the fact that U is an irreducible matrix of the class $\mathscr{D}_A$, we infer that

$$\eta = \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2,..., \int_{t_1}^{\tau_{l-2}} \times \prod_{s=0}^{l-2} \lambda_{i_{s+1}}^{i_s} u_{i_{s+1}}^{i_s}(\tau_{s+1})\, d\tau_{l-1} > 0,$$

where $l < \binom{n}{n-k} + 1$. This and (6.5) give

$$\lambda_j^i X_{F_{x_1,x_2}}\left(t_2; \frac{\beta}{\bar{\alpha}}\right) \geqslant \left\{\delta_j^i + \eta \exp 2(k-n)\binom{n}{n-k}\mu\right\} \exp(k-n)\,\mu.$$

Since

$$\det(I_\alpha^\alpha + I_\beta^{\bar{\alpha}} X_{F_{x_1,x_2}}(t_2)) = X_{F_{x_1,x_2}}\left(t_2; \frac{\beta}{\bar{\alpha}}\right),$$

therefore, by the last estimate, the inequality (6.1) holds with

$$\delta = \eta \exp\left\{(k-n)\,\mu\left(1 + 2\binom{n}{n-k}\right)\right\} > 0.$$

Thus the proof is finished.

Remark. If the hypotheses of Theorem 6.1 hold without (ii), then the assertion of Theorem 6.1 remains valid with $\beta = \bar{\alpha}$.

Indeed, if $\beta = \bar{\alpha}$, then by (6.5) we have

$$\lambda_i^i X_{F_{x_1,x_2}}\left(t_2; \frac{\bar{\alpha}}{\bar{\alpha}}\right) \geqslant \exp(k-n)\,\mu,$$

where $i = h_{\bar{\alpha}}$ and $\lambda_i^i = 1$. This means that for $\beta = \bar{\alpha}$ the condition (6.1) holds with $\delta = \exp(k-n)\,\mu > 0$.

We conclude this paper with an example illustrating the use of Theorem 6.1.

EXAMPLE 6.1. Let us consider the nth order differential equation

$$x^{(n)} = g(t, x), \qquad x \in R, \tag{6.6}$$

with the boundary value conditions

$$\begin{aligned} &x^{(i_1)}(t_1) = c^{i_1},..., x^{(i_k)}(t_1) = c^{i_k}, \qquad 0 \leqslant i_1 < \cdots < i_k \leqslant n-1, \\ &x^{(i)}(t_2) = c^i, \qquad i = 0, 1,..., n-1, i \neq i_1,..., i_k, \end{aligned} \tag{6.7}$$

where k is a given positive integer such that $1 \leqslant k \leqslant n-1$.

Assume that the function $g\colon J\times R\to R$ is continuous and has bounded continuous partial derivatives $g_x=\partial g/\partial x$. Assume, moreover, that for every $x\in C_1$, and every $t\in J$ the equality

$$(-1)^{n-k+1}g_x(t,x)\geqslant 0 \tag{6.8}$$

holds.

Using Theorem 6.1 and its remark we shall show that for every $\alpha=(i_1,\dots,i_k)\in G_{k,n}$ and every $(c^1,\dots,c^n)\in R^n$ the boundary value problem (6.6); (6.7) admits exactly one solution. In fact, the problem (6.6), (6.7) is equivelant to (5.1), (4.4) with $f(t,x)=(x^2,\dots,x^n,g(t,x^1))$ and $\alpha=(i_1,\dots,i_k)\in G_{k,n}$, $\beta=\bar{\alpha}$ and it remains to verify the condition (i) of Theorem 6.1.

For the Jacobian matrix $F(t,x)=(f^j_{x^i}(t,x))$ of f we have $f^j_{x^{j+1}}(t,x)=1$, $j=1,\dots,n-1$, $f^n_{x^1}(t,x)=g_{x^1}(t,x^1)$ and zero elsewhere. Note that all entries of $B_{n-k}(F^0)$ obtained from $f^j_{x^{j+1}}$, $j=1,\dots,n-1$, by (3.1) are equal to 1, whereas the entries of $B_{n-k}(F^0)$ from $f^n_{x^1}$ by (3.1) are equal to $(-1)^{n-k+1}f^n_{x^1}$. From (6.8), it follows that $B_{n-k}(F^0)$ belongs to $\mathscr{D}_\Lambda$ if Λ is defined by the vector $\lambda=(1,\dots,1)\in R^N$, $N=\binom{n}{n-k}$. Hence (i) is verified.

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